Teaching and Learning Group Structures in the Elementary School: an Experiment

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For some years, schools have been on the threshold of remarkable changes in elementary school mathematics, changes in *what* is being learned as well as *how* this learning is taking place. New mathematical topics have been introduced into the elementary classroom based on the belief that, at an early age, the child is able to grasp many of the important concepts previously reserved for more mature mathematical minds. Mathematics, as an hierarchical system of abstractions, imposes the need for accurately stated aims and objectives. What do teachers intend to achieve by requiring that children learn mathematics? Recent research in mathematics education has yielded new insights into the "whys" and the "hows" of learning mathematics, in that much of the value of mathematics lies in the *thinking skills* that a person acquires through the mental manipulation of mathematical concepts and abstractions. The current emphasis on thought processes as an aspect of mathematics instruction has been cogently expressed by Frank Land:

> . . . mathematics may be thought of as a highly disciplined mode of thinking. Many situations in the real world need to be thought about, assessed, appraised and criticized from the point of view which can be illuminated by mathematical thinking. By this I do not mean using mechanical computational skills, but the appreciation of the *structure* and pattern, which underlie them . . .

It is, therefore, the exploration of the structure of mathematics which forms the foundation of the learning of mathematics. It is difficult to separate what we term mathematical thinking and what is usually described as mathematical learning, especially since what is being learned is a set of concepts or structures.

There is an almost endless variety of structures with which we grapple so that we may survive in our technological society. In order to determine whether such structures can be *taught* and *learned*, experimental situations must be created in order to "discover" certain *predetermined* structures, and their related abstractions. For purposes of this experiment, the finite mathematical group structure was selected, and a controlled experimental study was undertaken at the Centre de Recherches en Psycho-mathématique, at the Université de Sherbrooke in Sherbrooke, Quebec. Two important questions were posed:

1. "Can children learn and abstract the structure of a mathematical group?"
2. "Does previous mathematical training affect the learning of mathematical group structures?"

Subjects

In order to have access to a wide range of children with exposure to different methods of instruction, 148 French-Canadian girls, from five different elementary schools and thirty-nine different classes, were selected: fifty-two fourth graders, fifty-three fifth graders and forty-three sixth graders. The sampling represented equal numbers of Cuisenaire, Dienes, and traditionally trained subjects. All subjects ranged in age from nine to twelve years and were assigned triads based on the same I.Q., socioeconomic status, age and previous exposure time to the same mathematical methodology.

Procedure

Since the participating children represented three different grade levels, three different mathematical group structures were developed. Special embodiments of the *Klein group*, the *Cyclic Eight group* and the *Cyclic Five group* were designed and prepared through concrete games, and systematically presented on individual task cards.* The experimental period was approximately eleven weeks in duration with 100 minutes per week of instructional time given by a Piaget-trained Swiss educator. During each instructional session, the child was led from an exploratory-manipulative period

*Examples of these cards are given in William E. Lamon and Lloyd F. Scott, "An Investigation of Structure in Elementary School Mathematics: Isomorphism," *Educational Studies in Mathematics*, Holland; Fall, 1970.

and a representational play period according to her own learning pace, through six distinct and ordered levels of mathematical thinking. Each level of learning embodied a different mathematical abstraction and manifested either at the pre-operational stage, operational stage or at the hypothetical-deductive stage of mental development. Those six levels were ordered as follows:

The first required recognition and understanding of the rules of the games. The second level of learning required the subjects to recognize the existence of an *isomorphic* relationship between different games having the same structure. In other words, the children had to demonstrate their awareness that in different games embodying the same structure, states and operators in one game correspond to definite states and operators in the other game. The third level of operation introduced the concept of binary operation; the fourth, the binary operation itself, required the abstraction of the concept of the particular group structure. At the fifth level of learning, the subjects became aware that if a true equation is transformed by an *automorphism*, another equation would emerge which holds true in the same system. The sixth level of abstraction was reserved for the discovery of the relationship between the *homomorphisms* and the group operation. In other words, the subjects discovered that two successive automorphisms can be replaced by a single automorphism. The level ordering was planned so that the subjects who did not reach a lower level, would not be capable of reaching higher levels.

Before a pupil was deemed ready to progress from one level to the next, she had to reach a behavioristic criterion of performance on a set of tasks. If the pupil succeeded in coping with a minimal number of the tasks, she progressed to the next level. Otherwise she was given an additional set of tasks. If the pupil failed to cope with the additional set, she was given still more tasks, until she was prepared to pass to the next level of learning.

Table 1 presents the distributions of the number of task cards, stimuli and number games by level of learning.

Findings

In order to indicate whether children can learn and abstract the structure of a mathematical group, each subject was given a score representing her highest level of performance achieved at the end of the experimental period. This score, with an assigned value from one through six, identified the highest level of problem complexity accomplished by each subject. Table 2 presents the mean scores.

TABLE I

DISTRIBUTION OF NUMBER OF TASK CARDS, STIMULI AND GAMES BY LEVEL OF LEARNING

Levels of Learning		*Number of Games*	*Number of Task Cards*	*Stimuli*	*Observation*
	L^1	g^1	6	130	Compulsory
		g^2	6		
		g^3	6	36	
		g^4	4	28	
	L^2	g^1 & g^2	5	58	Compulsory
		g^2 & g^3	5	50	
	L^3	g^1	6	96	Compulsory
		g^2	6		
KLEIN		g^3	6	54	
		g^4	5		
	L^4	g^1	12	154	Compulsory
		g^2	12		
	L^5	g^1	12	120	Compulsory
	L^6	g^1	10	41	Compulsory
	L^1	g^1	9	258	Compulsory
		g^2	8		
		g^3	7	137	
	L^2	g^1 & g^2	7	67	Compulsory
		g^2 & g^3	7	68	
	L^3	g^1	10	177	Compulsory
CYCLIC 8		g^2	9		
		g^3	8	84	
	L^4	g^1	11	128	Compulsory
		g^2	9		
		g^3	8	53	
	L^5	g^1	17	136	Compulsory
	L^6	g^1	7	28	Compulsory

TABLE I (continued)

DISTRIBUTION OF NUMBER OF TASK CARDS, STIMULI AND GAMES BY LEVEL OF LEARNING

Levels of Learning	Number of Games	Number of Task Cards	Stimuli	Observation
L^1	g^1	7	156	Compulsory
	g^2	6		
	g^3	6	130	
	g^4	5	20	
L^2	g^1 & g^2	7	58	Compulsory
	g^1 & g^3	7	58	
CYCLIC 5 L^3	g^1	7	116	Compulsory
	g^2	7		
	g^3	6	48	
L^4	g^1	11	122	Compulsory
	g^2	10		
	g^3	10	57	
L^5	g^1	11	70	Compulsory
L^6	g^1	8	38	Compulsory

In general, Table 2 demonstrates a pattern of increasing performance with age. At all three grade levels, the mean performance of the Dienes' subjects are higher than those of either the Cuisenaire or the traditional subjects. To assess whether program and/or age effects existed, the raw performance scores were treated as a randomized block (factorial) design. The analysis of variance results are presented in Table 3.

TABLE II

MEANS AND STANDARD DEVIATIONS OF PERFORMANCE LEVELS FOR EACH PROGRAM WITHIN GRADES FOUR, FIVE AND SIX

	DIENES			*CUISENAIRE*			*TRADITIONAL*		
Grades	*Mean Score*	*S.D.*	*n*	*Mean Score*	*S.D.*	*n*	*Mean Score*	*S.D.*	*n*
4	3.31	1.72	16	3.05	1.40	18	2.77	1.56	18
5	4.00	1.02	17	2.00	0.94	18	3.00	0.57	18
6	5.53	0.82	15	5.46	1.02	13	5.46	0.85	15

TABLE III

ANALYSIS OF VARIANCE OF THE EFFECTS OF AGE AND PROGRAM DIFFERENCES ON THE LEVEL OF PERFORMANCE SCORES

Source of Variation	*Degrees of Freedom*	*Mean Squares*	*F*	*Level of Significance*
Programs	2	7.869	5,398	.01
Grades	2	98.605	69.156	.01
Interaction	4	4.875	3.419	N.S.
Within Replicates	139	1.426		
Total	147			

The conclusions drawn, are:

1. For the hypothesis of no program effects, an F of approximately 4.79 is required for rejection at the .01 level; hence, as the obtained F is above this, it can be stated that a difference in mathemathical training results in a significant difference in performance.
2. For the hypothesis of no-age effects, the F value for grades of 69.15 is considerably in excess of the required 4.73 and hence, performance increases with age.
3. No significant interaction effects seem to exist between programs and grades.

As previously pointed out, the levels of learning were ranked in accordance with a hierarchy of difficulty and mathematical complexity. It was therefore important to assess what percentage of the participating subjects, by grade, attained each level of learning at the end of the experimental period, represented in Table 4.

TABLE IV

PERCENTAGE OF SUBJECTS BY GRADE ACHIEVING LEVELS 1-6 AS HIGHEST LEVELS OF LEARNING AT THE END OF THE EXPERIMENTAL PERIOD

Grade	*Levels of Learning*						*Total*
	L1	L2	L3	L4	L5	L6	
4	23	15	23	21	8	10	100
5	17	6	53	13	9	2	100
6	0	0	2	19	7	72	100

From the percentage distribution of the subjects it may be noted that 62% of the fourth graders, 77% of the fifth graders and 100% of the sixth graders achieved beyond the second level of learning. The passing of this level assumed the subjects' mastery of the structure of the game presented. The passing of the fourth level, however, assumed the mastery of the structure of the mathematical group. In this instance, the fifth graders seem to have encountered difficulties: 11% of the subjects were able to tackle the tasks required of them in order to progress to the next level of performance. At the fourth grade level, performance seems to be slightly higher; 18% of the participating children were able to pass the fourth level of learning. On the other hand, 79% of the sixth graders successfully passed the tests administered at this level. This result was not expected. It reflects the theory of mental development of a child as advanced by Jean Piaget: levels one and two embodied activities at the pre-operational stage; levels three and four presented experiences at the operational stage. The hypothetical-deductive stage was represented by levels five and six, which show a small number of failures at the sixth grade level.

Some Related Classroom Observations

With the exception of the sixth graders, the children had difficulty in handling the structure of the tasks presented to them at the first level of learning. When the fourth grade subjects encountered the first criterion test, 83% of the Cuisenaire children, 66% of the Dienes children and 61% of the Traditional subjects failed. Of those who failed, 73% of the Cuisenaire, 36% of the Traditional and 8% of the Dienes subjects failed the next criterion test. And again, of those who did not pass the second test, 25% of the Traditional sample failed, while all the Cuisenaire and Dienes children passed. It would seem then, that at the fourth grade level, the Dienes children exhibited a somewhat superior performance. The fifth grade children followed somewhat the same pattern of behavior. However, the Cuisenaire children here displayed a performance inferior to the other two groups: after the third criterion test, 89% of those who failed the second criterion test (i.e., 53% of those who failed the first criterion test), failed again. All subsequent levels of learning seemed to present the same difficulties to all subjects, irrespective of their previous type of mathematical education. It seemed obvious that when the end of the instructional period drew near, if the subjects at the fourth and fifth grades had been given additional learning time, the majority of them would have reached the sixth level of learning.

During the instructional periods themselves, attempts were made to reduce various kinds of extrinsic motivation, such as fear conditioning and/or social conditioning, which could have negatively affected the learning process. The establishment of intrinsic motivation through the use of the materials and task cards which had been developed, seemed to have highly motivated the children. During the eleven weeks of instruction, only three children out of 150 starting participants, had to be dismissed because of justified, but too frequent, absences. Furthermore, no escape strategies, such as requests for drinks of water, or use of bathroom facilities were used during the whole period of instruction.

Of special interest are the recorded observations of the classroom organization. At each instructional session, children were permitted to work together through the formation of small groups, each with a maximum of four children. Small group dynamics suggested different types of collaborative behavior in the different programs. Children in one group type had the tendency to borrow answers from the group leader without verifying the answer, while others discussed their problems with each other before putting down identical answers. The former type of behavior seemed to be more common in the Traditional and Cuisenaire groups, while the latter was more common to the Dienes group. This might be a consequence of the methods administered within the respective programs. However, the high percentage of those who copied meaninglessly in response to the tasks presented was unexpected.

Conclusions and Implications

The interpretation of the quantitative results, combined with classroom observations, seemed to indicate that if fourth, fifth or sixth graders are given a sufficient amount of learning time, they can learn and understand the idea of fundamental structure of the mathematical group. This statement does not suggest a formal representation of such a structure, but rather it indicates that if children are given concrete representations of the mathematical group, they can exhibit behaviorally the mathematical thinking and insightful learning associated with the study of the abstraction. It is difficult to reject the hypothesis that the type of mathematical education is related to the level of performance attained by the participating children, because each grade level was presented with a different mathematical group structure. Of importance to this study, however, was the tenet that the three group structures were equal with respect to *learning difficulty,* and were all developed and presented individually in the same fashion by the same teacher. Hence, differences in observed performance could be due, not to the type of mathematical learning, but to the type of mathematical structure presented. Evidence clearly indicates increasing levels of performance in the progression from the fourth to the sixth grade. In view of the superior performance of the sixth graders, the assumption that the Klein group is a difficult structure to handle, seems to be rejected. It could be argued, however, that because those subjects were advanced in the stage of mental development and older chronologically than the fourth and fifth graders. They may have had advantages that significantly affected the level performance.

A great deal of experimental work has been and is being done, to design improved classroom techniques and the understanding of ideas in elementary school mathematics programs. The present study suggests that the presentation of concrete examples of certain mathematical groups generates learning situations which will contribute to a real appreciation of the interconnections of the various processes which children must learn. During the experiment, very few children exhibited a lack of motivation in the handling of the structures. The more abstract they were, the better the children liked them. Hence, it seems that if we continued to stretch the child's natural desire to explore, just enough to make the learning of structures interesting for them (but not so difficult as to be impossible), we would perhaps have generated a still greater enthusiasm in mathematics learning.

Before the children seemed to be fully aware of the structure of a particular concrete example, the structure had to be presented in such a systematic way as to allow the passing from concrete manipulation, first without, then with verbalization, to symbolization, requiring a time span which varied as to the rate of learning by each individual child. An important implication resulting from this observation is that, in order for the process to lead to a meaningful abstraction, a precise hierarchy of mathematical pedagogy is required.

Symbolization was a critical factor in this research endeavor. The way in which the elements in the given groups were represented had a powerful effect on the rate of learning of each child. The experiences on which the concept had to be developed in order to become operational seem to require the introduction of the corresponding symbolizations. Learning, however, was not tied to a certain particular symbol system. In order to "engineer" insights into the roles of operators and states within the structures, symbol systems varied with the physical situations. The findings of this study would suggest that if children are required to manipulate on an operational level the structure of the group, a variety of symbol systems should be presented. Verbalization of an adult type should not be necessary, since children are able to perform the tasks without relying on verbal expression, i.e., a child will understand the structure when he is able to handle it.

Suggestions for Further Research

The criterion measures used in this study should be tried with a larger sample including both male and female subjects who have been exposed to either the same or different types of mathematical education. Attempts should be made to assess the effects of the learning of concrete examples of group structures upon the formal representations of the abstractions generated by the study of the theory of groups. It seems reasonable to assume that certain types of concrete examples should allow for a better understanding and acquisition of mathematical abstractions than others. It is suggested that embodiments should be developed which differ from each other to a greater extent than those presented in this study. Another suggestion would be to present the "same" simple or complex structure to children of different grade levels to establish a more accurate study of the effects of age upon the learning of group structures. Such a presentation would be more amenable to analysis by the use of statistical techniques appropriate to comparative studies. The test items could then be more carefully constructed so that the measured behavior of the children would be more appropriate to the grade levels considered.

Similar problems in structures other than groups might also be profitably investigated. The learning of relational concepts such as equivalence and order could be subjected to experimental treatment like the one in this study. Variation in the structures used in future studies is considered essential if any general formulation is to emerge to show how structures are learned.

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