SOME ASPECTS OF THE LEARNING OF THE LANGUAGE OF MATHEMATICS

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To speak of the "language" of mathematics is to use the word "language" in a special sense. The French, in speaking of language in the sense of "tongue" use the word *langue,* and for the other use, more appropriate here, they use the word *langage.* There is a language of mathematics in this latter sense, with a special vocabulary and special symbols. Also, one might say that the language of mathematicians is distinguishable from the language of mathematics. This article attempts to deal with some of the peculiar properties of the language of mathematics, and to offer to both students and teachers of mathematics some suggestions for acquiring more insight into mathematics by an examination of the language of mathematics.

Mathematics Contains Abstract Entities

The entities of mathematics are essentially abstract. These entities have names and they are usually represented by particular symbols. In general, two different mathematicians, when referring to the same entity, will use the same symbols, but may call them by different names. To both a French-speaking and an English-speaking mathematician, the symbolic form $\{ x \mid x > 5, x \in R \}$ will represent the same entity, but the verbal form will not be the same. To the French-speaking mathematician, the symbols represent what he would call "l'ensemble de tous les x tels que x est plus grand que cinq, et x est un nombre réel," while his English-speaking counterpart would say "the set of all x's such that x is greater than five, and x is a real number." In both cases, the object of thought is the same; so is the symbolic form.

The Growth and Complexity of Mathematical Symbols

In most cases, mathematical symbols are quite international, giving to mathematics something of the characteristic of music, often thought of as an international language. However, the universality of the language of mathematics is somewhat illusory. In dealing with mathematical processes, mathematicians have to talk about the objects of thought through verbal forms which are not purely symbolic. It is in this area of verbal expression that certain difficulties arise, even among mathematicians who speak the same tongue. There are enough variations in verbal descriptions to cause ambiguities. Hence, the layman who presupposes that the language of mathematics is clear and completely unambiguous may be off to a false start.

This is not vitally serious, however, if a proper attitude is developed. For the most part, the symbolic representations of mathematical entities *are* definitive and unambiguous. Also, mathematicians are constantly striving to improve both symbols and vocabulary so that ambiguity will disappear. The language and symbols of mathematics, like that of English, or any other tongue, are subject to the effects of tradition, incomplete knowledge, and prejudice. The symbol " $+$ ", to indicate addition, had to face and overcome opposition, since its use seemed to be sacreligious. The names "real" and "imaginary," to describe certain sets of numbers, are traditional, devised at a time when the true meaning of number was imperfectly understood. What does 2 . 3 mean? Is it two and three-tenths, or is it the product of two and three? To most Europeans, 2,456 is a symbol referring to a number between two and three, whereas to most North Americans, it represents a number one thousand times as large. Newton of England and Leibniz of Germany discovered the calculus at about the same time. For the function defined by $y = x^2$, Newton symbolized the corresponding derivative by $\dot{y}=2x$, and Leibniz used $dy/dx=2x$. Out of national pride, English mathematicians insisted on the dot for a good hundred years or more. At the same time, the mathematicians of Europe were using the ratio form. It has been conjectured that one possible reason for the dearth of first rank mathematicians in England during the eighteenth century was the artificial barrier to communication with other mathematicians set up by the refusal, on patriotic grounds, to have anything to do with the Leibniz notation. This problem was resolved early in the nineteenth century, and that century produced such British titans as Hamilton, Sylvester, and Cayley.

The Language of Mathematics Controlled by Experts

Unlike languages spoken by ordinary people, the language of mathematics is normally pretty well under the control of experts. This is particularly true today. It is sometimes said that the man who invents the best set of symbols is the one most likely to be read and understood. Of course, the influence of publishers and printers is of great importance in the dissemination of symbolic forms, as it is in standard languages in the matter of spelling and grammar. One of the most discouraging aspects of so-called "modern" mathematics to the traditionally trained teacher is the vast array of new symbols and names to be learned.

To some, it may seem that the proponents of the new mathematics are making a fetish of symbols. In a few instances, this supposition may not be far from the truth. But, by and large, what is really being attempted is the organization of symbols for the purpose of achieving more clarity, succinctness and simplicity. However, the resulting plethora of new symbols to be learned is often frightening to the tradition-bound student, much as the introduction of reformed spelling would be to most traditionally trained users of English.

One simple example of the attempt to clarify symbols is found in the modern usages of the symbol "–", almost invariably called "minus" in any context in the past. While the symbol is still kept, the term "minus" is used when subtraction is indicated, and the term "negative" is used when the symbol refers to the sense of the number. Even here, there is some confusion, requiring such precision of expression that correct usage becomes almost pedantic. The numeral "-7" would be read "negative seven", while "$-x$" would be read "the negative of x", since $-x$ is not necessarily a negative number. In Algebra, $5-2$ can be thought of a $^{+}5-{}^{+}2$, indicating the subtraction of two positive integers, or as $^{+}5 + {}^{-}2$, indicating the sum of a positive and negative integer. The two are not identical, but are mathematically equivalent. In ordinary arithmetic, $5-2$ unambiguously indicates a subtraction.

To the serious mathematician, interested in communicating ideas, these niceties are important — moreover, they are aesthetically attractive, for, as literature is an art, requiring language to express it, so to many, mathematics is also an art, and elegance of expression is sought after. However, for lay use, practical considerations often force aesthetic considerations into the background, and the finest distinctions are not, and need not, be made.

Should the Language of Mathematics be Approached as a Foreign Language?

Those who teach mathematics, and those who learn it, must concern themselves with two things: the content of mathematics and the use of it. Often, those who say that they cannot learn mathematics have not approached it properly, and this could be the result of inadequate teaching. Too often, the same approach is used in the teaching of mathematics as is used in the teaching of English to English-speaking students. It should be remembered, however, that the student of English is studying that with which he has considerable familiarity, and which he uses, more or less effectively, all the time. Although the words used in the teaching of mathematics may be English words, these words are technical, and the symbols are specialized. Thus, the teaching of mathematics may require techniques more in line with the teaching of a foreign language than those of teaching the mother tongue. This means that the correct interpretation of symbols, acquisition of vocabulary, and other attendant skills should become an integral part of the teaching of mathematics.

The Teaching of the Language of Mathematics Must Begin Early

Present methods of teaching mathematics are closely linked with what is called the "discovery" method. This is of great importance to educational psychologists, but perhaps is not germane to the present discussion. However, the intercommunication between pupil and teacher in the discovery method is an aspect of language, and the language would normally be the mother tongue, with an incidental use of mathematical terminology. Once the discovery has been made, the degree of perception is measured in terms of correct use of mathematical expression. In the teaching of division of fractions, for example, the question "What happens to this number if I divide it by a number that keeps getting smaller and smaller?" would be asked in the vernacular, and first answered in the vernacular. The final generalization would be required in mathematical language, such as: "To divide a number by a fraction, multiply that number by the reciprocal of the fraction; the result is the required quotient." Later, the symbolic form $(p \div a/b = c) \leftrightarrow (p(b/a) = c)$ would be introduced.

It may well be that in the beginning stages, content will precede terminology. Thus, a child on Cuisenaire rods may find by experiment that rod color A, placed end to end with rod color B, yields a total length the same as that of rod color C. At this point he will have only the colors to describe his discovery Later, he may see that the 3-rod and the 4-rod have the same combined length as the 7-rod. But eventually he has to be able to say "three and four are seven", and write "$3 + 4 = 7$". Further, when he has reached the stage of writing numerals, he may forget the sentence $3 + 4 = 7$", and when confronted with the incomplete sentence "$3 + 4 = -$", may have to use his rods. But this would be ineffectual unless he recognized the meaning of the symbols in the incomplete sentence. His failure to provide the necessary "7" would not be interpreted as lack of knowledge of the language, but a failure in memory or association of some kind. He has failed, perhaps for lack of sufficient drill or understanding, to associate "$3 + 4$" with "7". If he should write "12" instead of "7", one might conclude that he had confused "x" and "+", and this would be a mistake in vocabulary.

While one should not overburden young children in the elementary school with too many symbols, there must be some use made of the standard symbolic forms, and these should be taught as essentials of the course. There is no harm, and much value, in introducing general symbols like "n" into an arithmetic course. This is being widely done in most modern methodology and seems to be successful. If the child learns some symbols beyond the usual numerals and operational signs in the elementary grades, he is less likely to be overwhelmed by the sudden need to master a great many symbols when he reaches high school.

One could teach arithmetic with little or no generalized system of symbols. It may be possible to teach algebra as one teaches arithmetic. After all, the algebra of the Middle Ages contained expressions such as "res et tres", where we use "$x + 3$". But the vast amount of mathematics now developed would have remained unknown were it not for the clarity and succinctness of the sophisticated system of symbols and terminology so painstakingly assembled during the past four hundred years. Moreover, at a certain stage of mathematical development, the symbols seem to transcend their referents and become in themselves objects of thought.

Role of the Teacher in the Learning of the Language of Mathematics

The teacher of mathematics, being aware of the importance of the symbols and special vocabulary of mathematics, will realize that they must be carefully taught if his students are to learn mathematics properly. Any ambiguities in symbols or terminology can be overcome quite readily by sticking initially to some standard forms, and using them consistently. If this is done, the student can concentrate on the content. Later, when the student is more mature, he will be able to read without being baffled works in which other notations and terminology are used, provided that the author, as he should, explains his terms carefully as he introduces them.

Although the teacher may treat the learning of the language of mathematics in a manner analogous to that of teaching a foreign language, he cannot carry the analogy very far. At the elementary level (kindergarten through college) mathematics is concerned with symbolic representations of assertive sentences combined by various connective devices into more elaborate structures, but avoiding, in general, commands, questions and exclamations. Further, the "verbs" of mathematics are usually expressed through binary relations and are not numerous. Refinements such as case, person, mood, tense, of traditional grammar, are not used. The general schema of a mathecatical sentence with two variables is yRx (y is related to x) where R is a relation. Or there may be several variables involved, and a more flexible schema is R(x,y,z . . .). An example of the use of yRx is $y = 2x$, x being the independent variable and y the dependent variable, related so that y is twice x. If R means "is greater than the square of", yRx means $y > x^2$. In the case of $4 + 5 = 9$, $4 + 5$ is an instance of y and 9 is an instance of x and R means "=". Most of the relations studied in the elementary stages are binary in nature, relating two things at a time, whereas in English, one has instances like "John gave Betty a watch for her birthday," where "gave" relates four entities.

Even though the language of mathematics is not truly the same in structure as a foreign tongue, the teaching procedures follow similar trends in emphasis on pattern. In ordinary language, an assertive sentence has a certain structure: article, adjective(s), noun-subject (or pronoun), adjectival phrase (or clause), verb, verb modifiers, objects, object modifiers, and so on. Many of these variables are removable, but to be a sentence, the schema requires a verb. The language of mathematics is not so flexible, and wide variations in form are not standard.

Because of less variation in form, it should follow that the learning of the language of mathematics will offer less difficulty to the beginner than the learning of a foreign tongue. The areas of difficulty will be in vocabulary and translation, not in sentence form or idiom. One of the common errors made by teachers of mathematics is lack of emphasis on the precise meaning of terms and symbols. Many students have difficulty in formulating a mathematical sentence properly representative of the verbal sentence that has been given. The teacher should make sure that the trouble does not lie in lack of knowledge of the symbols.

The language of mathematics is not a natural one — it has been created to serve the needs of mathematics — so it must be learned, and it must be taught. Lack of familiarity with the language of mathematics does not brand one as generally illiterate, only mathematically illiterate. A famous, but apocryphal, story illustrating that mathematicians have a special language concerns Euler's reply to Diderot. The story is that Diderot, when at the Russian court, was making everyone extremely uncomfortable by his agnostic arguments about the existence or non-existence of God. None of the court philosophers could counter his statements. Euler, the great Swiss mathematician, and a devout Christian, was also at the court. He was asked if he could offer any remark to confound the invincible Diderot. He thereupon came forward and said: "Monsieur, $x^n + px + q = 0$; donc, Dieu existe." Diderot, unfamiliar with the language of mathematics, had no way to answer him.

So the story goes. It is most improbable, however, since Euler was no charlatan, and Diderot was no fool.

Knowledge of the Language of Mathematics Necessary but not Sufficient

Familiarity with the language and notation of mathematics is one thing (careful teaching and conscientious study can accomplish this for most pupils) but it does not necessarily follow that one who knows the language can do mathematics. Mathematics requires that the properly notated statements be combined in a logical framework to produce a certain conclusion. To be able to do this, the learner must be able to reason in logical sequence, and must have an eye for pattern. The teacher must be able to relate what is being done to the conditions prevailing — the pupil's background, intelligence and experience.

Most difficulties experienced by children in learning mathematics are not based primarily on the learning of symbols and terminology. It is in the logical framework, where patterns and implications must be recognized, that the greatest difficulty is found. This is, of course, a formidable matter, and requires that the teachers of mathematics, at any level, thoroughly understand the conceptual aspects involved. But all efforts to break down concepts into fine fragments, so that all steps can be completely followed, will fail unless there is some clear, concise, reasonable way to express these concepts. As the scope and variety of the content increase, there will be more and more need to rely on full familiarity with the language of mathematical thought, and eventually, much of this will be the use of purely symbolic forms. One cannot learn mathematics effectively merely by familiarity with the language of mathematics; familiarity with the language is a necessary, but not sufficient condition for the learning of mathematics. Thus, to acquire any real power over mathematics, one must acquire power over the symbols, and these, like the patterns of a foreign tongue, must be learned thoroughly as they come along — not all at once — but as the need for them arises.

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THE ENVIRONMENTAL STUDIES PROGRAMME
ANOTHER ASPECT OF ACTIVIST TEACHING AT McGILL'S FACULTY OF EDUCATION

In the Spring of 1966, the Department of Instruction in Geography and History at McGill's Faculty of Education, attempted to introduce groups of prospective teachers to environmental studies as an aspect of activist teaching method. Taking advantage of the Sunday traffic lull in Montreal, parties of student teachers and their instructors explored the geographical and geological structure of Mount Royal, and attempted an on-the-spot investigation into some of the historical and geographical principles influencing the growth of the old city itself. These expeditions were not conducted tours, but demonstrations of how teachers might guide children through a systematic, if informal, exploration of the local environment.

To illustrate further the possibility of this kind of activist teaching, the Department, in close association with the Macdonald Elementary School and the Faculty of Education's Audio-Visual Centre, worked with a Grade VI class in a similar project. On this occasion the geographic and historical features of Ile Perrot were substituted for the conventional textbook, and the children, using the inductive method where possible, discovered for themselves, meaning in the local environment. The results were recorded in a modest but interesting colour film.

Encouraged by the success of these early attempts, a number of Departments of the Division of Curriculum and Instruction plan to cooperate in developing an Environmental Studies Programme as a regular part of teacher education at Macdonald College in the 1966-1967 session.

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