THE LEARNING PROCESS AND MATHEMATICS INSTRUCTION

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Introduction

In recent years much has been written concerning the widening gap between the newer developments in mathematics and the traditional mathematics taught in secondary schools. Not unnaturally, leading scholars in mathematics have looked at the school programmes and found them wanting. These people have considered that mathematics curricula should be reformed so as to bring the subject matter in line with recent developments. Professional educators, many of whom are prepared to take an active part in shaping the future policies of school programmes in mathematics, find themselves faced with the problem of translating the suggested reforms into action within classrooms. Thus, while it is recognized that the mathematicians are in the best position to determine the structure of the discipline, it is the professional educators' responsibility to prepare good school programmes in mathematics.

In the past, the main difficulty was to obtain for mathematics a legitimate position in the scheme of teaching. Today, this problem has been overcome, but another has arisen in its stead. The difficulty which now confronts those who are interested in mathematics education, whether from the point of view of general organization of the curriculum or in the actual teaching of individual branches of mathematics, is that of finding some guiding principles to animate mathematics teaching as a whole, to coordinate the instruction a student receives in different years in different branches of mathematics, and to decide which of the innumerable facts of mathematics should be inserted, which should be omitted.

If it is the mathematical point of view and not merely a collection of facts that we wish to impress on those we teach, then it becomes increasingly necessary to eliminate needless detail, to concentrate on fundamentals, to arouse the interest of the student at the outset. A student who has become involved in the ideas of mathematics and has been brought to appreciate mathematical method is educated in a much more desirable, and indeed in a much more complete, way than one who has succeeded in assimilating a large number of detailed facts.

There is a vast amount of mathematics already, and the explosion of knowledge indicates that there will be more. The rapidity of change means that it is not clear just what the specific applications and methods will be in future years. Therefore, it is important that we try to develop in our students a point of view which will enable them to learn additional mathematics. This point of view can best be achieved by allowing them to take an active part in the formulation of ideas and in the evaluation of material as it comes to them.

Traditional Mathematics

For some time now students have been exposed mostly to a one-way type of communication. They have text books that purport to "show them how to do a problem," and they are directed to solve particular types of problems — problems that have already been solved. Many teachers today leave their students with the impression that mathematics consists of a series of operations or "manipulative tricks" and if one follows the leader — teacher or textbook — the answers will appear. Textbook problems usually fit into a recognizable slot. A turning of a crank grinds out the answer. Thus every "problem" presented in school is in reality no problem at all. Any student who applies himself assiduously to such "problems" has composed nothing that might be an inspiration for his leisure hours.

Our best trained and most conscientious teachers of mathematics find themselves dominated by textbooks. They forego the benefits of extensive questioning and inquiry because the present situation does not permit them to deal with each student simultaneously. Parenthetically, "subject promotion" has brought about much homogeneity within groups of students, but our teaching is still directed at the average student. Of more importance, however, is the fact that in a dynamic and rapidly changing world, where the concept of change should be paramount, we find ourselves reduced to mediocrity because we cannot or will not allow students to use and test various strategies in reaching the final goal. We tell students, in the same way as we would tell a computing machine, that "this is the way the problem should be done." Only by a fortuitous arrangement of albumen and hydro-carbons do we get some of our students to respond. We discourage creative thinking, the intriguing phenomenon of choice does not occur and the student relies heavily on his ability to memorize. Now memory is a very useful intellectual power but many people who may be capable of absorbing an enormous amount of mathematical stuff and can recall it when necessary, may falter in the face of uncertainty of choice. It is this uncertainty that inhibits active, creative thinking and restricts both the learner and the teacher to considering only how the author of the textbook has suggested a problem *might* be solved. Thus we find in the mathematics classroom a group of people merely "learning about" something. Rather than engage these students actively in the task of discovering something new, in putting things together for themselves, we deny them the right to think.

Beyond the Bounds of Syllabus

Before proceeding, it may be worth while to note that school grades are not ad hoc groups but rather, they are dynamic groups concerned naturally with their own ideas as well as ideas of others. Through verbal intercourse and exchange of thought, a variety of responses is produced and the learner discovers the cunning of others. He becomes a strategist in gathering information. He is more likely to break down the barrier of compulsivity, to become more flexible, to discard a meticulous preoccupation with ideas that have already failed. One example of this can be observed in a game called "Twenty Questions" where, by an elimination process. a group of people, through questioning, approaches an answer as rapidly as possible. One might say that learning has been speeded up, because "the thinking of the learner" has been channeled "in such a way that the extreme incorrect hypotheses he may try out are eliminated from consideration."¹

Sometimes a student may suggest an alternative method for solving a problem. In this event he should be given an opportunity to demonstrate his ideas. He may find a more or less efficient method than the one prescribed by his text. The teacher, of course, should be prepared to point out the soundness or the fallacy in the reasoning process. He should be prepared to state whether the student has made a conjecture or discovered a generalization. In either case, by going beyond the requirements of a syllabus prescribed by the textbook, the student will have added to his knowledge. Again, when a teacher guides his students in solving a system of linear equations by the "triangulation sweep-out procedure," he will have introduced an idea directly related to the solving of linear equations by modern automatic digital computers. The identification of a concrete example motivates the student by bringing to him one of the many new and challenging ideas in the field of mathematics. One might say to the student that this process is tedious by pencil and paper, and let it go at that. But surely teachers of mathematics will agree that computers are likely to play an important part in our lives. Allowing the student to think beyond the minimum requirements of a syllabus would be in the finest tradition of mathematics, and consistent with the modern world, for we shall have, in this case, channelled the student toward thinking about the impact of computing machines on contemporary society.

Further, if a student has acquired knowledge concerning the geometry of Descartes then he should be permitted to synthesize an idea by combining this knowledge with his understanding of Euclidean geometry. This combining of ideas exemplifies the manner in which mathematicians work. They reach back into mathematical history, exploiting the reservoir of pure mathematics, thus making use of previous discoveries and, having investigated a topic, they pursue the inquiry, just as Albert Einstein did when he made use of tensor analysis which had been developed in turn from the geometry of G. F. B. Riemann.

These instances emphasize the value of designing a technique to utilize things that already exist in the physical environment, and to develop a more cooperative position in the classroom in regard to generating a solid basis of knowledge in the individual. As Piaget has noted ". . . without interchange of thought and cooperation with others the individual would never come to group his operations into a coherent whole: in this sense, therefore, operational grouping presupposes social life."²

From Intuitive Perception to Logical Reasoning

A number of researchers including Bruner and Dienes, and recently Allendoerfer, have suggested that an individual "first gets an intuitive perception" not based on logical thought or reasoning and "this rather vague perception urges him on to constructive or creative effort to confirm the intuition by logical argument."³ If we fail to recognize that constructive thinking develops before analytic thinking, we shall not derive the most profit from our efforts to engage the students in inquiry.

Consider the following. A student may observe that: 1 + 8 + 27 + 64 = 100

Helping him to develop this observation, we seek a pattern and find that:

$$1^3 + 2^3 + 3^3 + 4^3 = 10^2$$

Let us see if we can develop this idea further. We have:

1 = 1 = 1² = 1² 1 + 8 = 9 = 3² = (1 + 2)²1 + 8 + 27 = 36 = 6² = (1 + 2 + 3)²

This pattern seems to indicate that:

 $1^3 + 2^3 + 3^3 = (1 + 2 + 3)^2$

A student may suggest that: The sum of the first few n cubes results in a squared quantity. This tentative generalization, the teacher will know, is indeed so for all positive integers. We can generalize as follows:

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1^{3} + 2^{3} + 3^{3} + \ldots + n^{3} = (1 + 2 + 3 + \ldots + n)^{2}
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This example of a pattern of reasoning leads to a fundamental process in mathematics usually called "mathematical induction" which we can use when we want a rigorous proof or systematic deduction. The important thing here is to engage the student in arriving at the generalization before attempting the rigorous proof.

The teacher's role in this "guided discovery" process or "hypothetical mode," as Bruner has called it, is not merely one in which he exposes his students to certain pieces of mathematical knowledge, but it is rather a dual role in which the teacher must remain alert at all times to the underlying concepts of mathematics and to the principle of action or, as Gesell has termed this, "the principle of motor priority." "This principle," says Gesell, "is so fundamental that virtually all behaviour ontogenetically has a motor origin and aspect." ⁴ The mere exposition of knowledge is not enough, and the

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teacher should be prepared to listen and test novel ideas that emanate from the students themselves. Hilgard has noted,

In order to tolerate the frustrations along the way, prior to the thrill and excitement of discovery or creation, a student has to develop confidence in his own capacities as a creative person. There is no short way of engaging in inquiry and in creation, and receiving the rewards that come through creative efforts, for this confidence to be achieved. The teacher helps by celebrating small achievements, so that larger ones can come in due time. ⁵

Whether we regard mathematics from the utilitarian standpoint, according to which the student is to gain expertise in using a tool, or from the purely logical aspect, according to which he is to gain skill in argument, it seems clear that teachers of mathematics have not done enough either in helping students to converge upon a standard mathematical proof or in encouraging divergent thinking of students so that they may build, or structure, or create for themselves. These activities have been neglected, and too much attention has been given to characterizing definitions and designating them by symbols before emphasizing the awareness of a concept. The instructional practices in the schools have been inadequate because the usual procedures of chalking on blackboards and the making of marks in workbooks are not functional activities. Much of this sort of thing should give way to allow students the opportunity of making mathematical discoveries. and of proving or disproving conjectures. For example, if a student thinks he has discovered something, anything at all about prime numbers, then he should be allowed to explore his conjecture. He may write his name indelibly in the history of mathematics. If his discovery is already known to others, this in no way need detract from his accomplishment. He may not find a "black rose" but he may find a "black swan." In any event, the teacher is afforded an opportunity to acquaint students with Fermat's theorem of 1640. Goldbach's "guess" of 1742, Vinogradov's proof of 1937 and the "eurekas" of others who have tried to unlock the many secrets of primes.

In summary, the teacher who affords himself the privilege of participating with his students in "beating around the bush" for a while, even though concealing his own expertise, identifies himself as a person of high professional ideals, for he is joining his own students as "frontier thinkers" in mathematics education. In fact, the teacher who is likely to be successful in the future, is the one who gives serious consideration to the psychological foundations of the "action basis" for learning and is abreast of recent developments in mathematics. He will be both dreamer and practical man within the same skin.

An Approach to Creativity

All of the above implies that the teacher of mathematics must be well qualified to create a climate of learning. The point here is that, in securing an atmosphere more akin to that of pure scientific investigation, it will be necessary for the teacher to ensure that the student follows some kind of productive procedure. If this is not done, the important process of inquiry will not be mastered and we shall have in its stead a series of random trials, a "pot-shotting" procedure where errors are not evaluated, and where the student cannot know whether or not his responses are leading him on to the final goal. It may well be that no one teacher can control each student's behaviour. B. F. Skinner has warned us "that a teacher cannot supervise 10.000 to 15.000 responses made by each pupil per year." ⁶ Thus it may be essential that we turn our attention more toward teaching by television, team-teaching, and toward some of the "techniques for controlling behaviour, so that the student does actually go through more productive processes . . . and here some of the techniques of programming for guiding and controlling behaviour may be applicable." 7

In pointing out the need for a systematic approach to creativity, we cannot overlook the sort of training given to potential teachers of mathematics. Polya has noted that the curriculum of mathematics teachers has given no attention to the ability to reason and to creative thinking. Says Polya, "Here is... the worst gap in the present preparation of high school mathematics teachers." ^s This assertion by a renowned teacher of mathematics emphasizes the importance of methodology in the training of teachers. If we are to encourage young people to investigate a topic in mathematics as a voyage of discovery, we should pursue the quest, as it were,

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by engaging the intending teacher in the demands of the educative process. To this end, professors of mathematics education should have freedom to shape the methodology of their students in those lectures which have usually been devoted exclusively to subject matter, and this will mean, in turn, that examinations would be different, but by no means weaker, from those given in purely academic courses. Method is what counts when training teachers for primary and secondary schools. The inspiration must come from the teacher and not from the things that are taught —

... let us always recognize that content and method are exactly the things with which our subject is concerned; for *Mathemos* is literally 'A subject of Instruction' and *Mathematike* is 'The Art of Teachable Knowledge'. These are the two elements that make for us our profession.⁹

Another important factor concerned with a systematic approach to creativity, and one which should engage the minds of professors of education and student teachers alike, is the age at which the attempt can best be made to maximize active thinking. Improvements in the mathematics programmes of schools depend, to a great extent, upon the answers to such questions as "What about the age placement of geometry?" and "Generally what information can we extract from the development studies that will help us to structure curricula content in terms of the normative developmental time table of young people?" For example, the "ontogenetic" order of spatial concepts is first topological, followed by projective and Euclidian, with one exception: the abstract notion of infinity follows after the categorization of the aforementioned three concepts. ¹⁰ If we believe that "activist education" will enhance the teaching-learning process, then it behooves us as teachers to arrange the subject matter in a sequence which will correspond, as far as possible, to the cognitive development level of the student.

Thus it will be the teacher's task, and one often demanding considerable ingenuity, to analyze the content to be learned in terms of the operations implicit in it. Having done this, he will arrange the learning materials so that these operations can actually be carried out by the student himself, and then see to it that the student does carry them out.¹¹

In particular we should attempt to make fruitful use of the developmental findings of Piaget and others engaged in this particular branch of educational research. Mining the educational lode of such researchers will guard us against redundancy when planning curricula. An awareness of the most favourable time to cultivate the mathematical imagination of young people could provide us with the best conditions for increasing their productive creativity.

Conclusion and Some Recommendations

The studies of schoolmen will be atrophied if attempts are made merely to replace the deadwood of mathematics curricula by so-called "new mathematics." What is required is a bold attempt to engage students in "active commerce" within a world of mathematics, and this can best be achieved by allowing students to use their intuition more and to organize material in terms of their own interests.

Such an approach will necessitate that teachers be up-todate not only in subject-matter knowledge, but that they apply their pedagogical skills to organize information in a manner that will not discourage or confuse students. Moreover, changing over from a traditional expository form of instruction to one where we lead the pupil to be an autonomous and selfpropelled thinker, will involve us much more than in the past in a question-and-answer technique. We shall be catapulted, as it were, into the briar patch of an articulate and opinionforming society of unsophisticated mathematicians. Consequently, teachers will be compelled to look for other techniques for guiding and controlling behaviour, and to recognize the most favourable time in a learner's life when to apply an "activist technique."

Some areas for study in the teaching of additional mathematics include:

- i) programmed instruction, not just as a means of teaching mathematical "stuff," but also as a means of enhancing the "investigative process;"
- ii) the changing of examinations, particularly at the matriculation level, so as to emphasize more the importance of cognitive activity — arranging for

a student to take an examination when the teacher feels he is ready;

iii) a major overhaul in the training of teachers of mathematics, particularly those teachers preparing for primary schools — emphasis to be placed on the "growing body of knowledge of how young children learn mathematics."

The Royal Commission on Education in the Province of Quebec has demonstrated an awareness of the increasingly important role of mathematics in the contemporary world by indicating that any new programmes for school mathematics are now the responsibility of those actively engaged in education. We thus have the opportunity to exercise fundamental leadership in the creation of a better mathematics programme. It would indeed be tragic if we should not be equal to the task.

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