ADDRESSING PRE-SERVICE TEACHERS' MATHEMATICAL REASONING THROUGH REVERSE ENGINEERING

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ABSTRACT. In this Note from the Field, I show how undergraduate pre-service teachers exercised mathematical reasoning when they were required to find the rules determining their score in successive rounds of an iterative game. The rules were hidden from them, and therefore they needed to use mathematical reasoning to reverse engineer the rules based on their scores. The teachers generated similar conjectures as they worked to decipher the rules, even as, with each iteration of the game, pre-service teachers learned the rules to maximize their scores. Reverse engineering, as a pedagogical strategy, would seem to offer a promising avenue for teaching mathematical reasoning in teachers — who can then teach their students.

ABORDER LE RAISONNEMENT MATHÉMATIQUE DES FUTUR.E.S ENSEIGNANT.E.S PAR L'INGÉNIERIE INVERSE

RÉSUMÉ. Dans cette note de terrain, je montre comment les étudiant.e.s en formation à l'enseignement faisaient preuve de raisonnement mathématique lorsqu'ils/elles devaient trouver les règles déterminant leur score lors des manches successives d'un jeu itératif. Les règles leur étaient cachées, et ils/elles devaient donc utiliser le raisonnement mathématique pour rétro-concevoir les règles à partir de leurs scores. Les enseignant.e.s ont formulé des conjectures similaires en travaillant à déchiffrer les règles, même si, à chaque itération du jeu, les étudiant.e.s apprenaient les règles pour maximiser leurs scores. L'ingénierie inverse, en tant que stratégie pédagogique, semble offrir une voie prometteuse pour enseigner le raisonnement mathématique aux enseignant.e.s – qui pourront par la suite l'enseigner à leurs élèves.

I recently began teaching an in-person course on mathematical content knowledge for pre-service teachers. After some reflection about exactly which topics should be included and the order in which I should teach them, I decided to begin the first class with mathematical reasoning (MR). At the same time, I had to think about what I really believed MR to be. I wrote myself a rudimentary definition based on my experiences as a K-12 teacher and then, to my initial dismay, upon double-checking with the literature, discovered that the lack of agreement about MR has carried on from at least 1999 (Steen, 1999). Jeannotte and Kieran's (2017) more recent work reiterated the absence of consensus around MR. I drew some comfort from the fact that they had also created a framework for MR, which included: 1) a process component that accounted for student behaviors of searching for similarities and differences, validating, and exemplifying, and 2) a structural component. The structural component consisted of inductive (i.e., infer the warrant from the claim and data), deductive (i.e., infer the claim from warrant and data), and abductive (i.e., infer the data from the claim and warrant or by using the claim, infer the data and warrant) steps[1]. Encouraged, I ensured that my initial activity would include opportunities for students to showcase these structural and process components.

I then began to consider my options. Not only did I need to create opportunities for students to demonstrate these dual elements of MR, but I needed something interactive that would allow students to socialize with the other members of the class. After some brainstorming of several instructional strategies, I settled on having the students play a game rather than lecturing, or having them discuss what they think MR is, or why it is important to mathematics education. I would have the students utilize their MR skills in small groups as part of the game, finishing with each student constructing their own understanding of the structural and process components of MR.

I next thought about which game I might select, as I needed something that was accessible for students of varying skill levels. This would be the first class of a course including first year university students, so I did not want to scare them with anything too complicated or bore them with detailed semantics about mathematical reasoning. Teaching math can be challenging, particularly if students have had poor experiences with it before, so utilizing a game to ease students into the course structure seemed like a good fit. I set off down the game path.

Game Criteria

Returning to Jeannotte and Kieran's (2017) work, I reflected on the deductive, inductive, and abductive steps that constituted the structural component of MR. Sprague (2022) has identified the deductive component and referenced comparisons, validations, and justifications as also crucial to MR. In my previous decade of teaching K-12 (mostly science and mathematics), I had observed students hunting for patterns, organizing and categorizing information. For example, by guessing (i.e., establishing hypotheses), checking (i.e., testing hypotheses), and changing (i.e., revising hypotheses) as they completed complex problems. I recalled Yankelewitz (2009), who had postulated that the "ability to convince others through argumentation and justification forms the foundation of mathematical reasoning" (p. 17). I realized that the core part of the game needed to provide an environment for students to make certain testable claims, based on different inferences and evidence, towards either the deductive, inductive, or abductive step necessary to achieve the structural component of MR.

Students would need to search for similarities and differences, validate, or exemplify the relationships they discovered. The best thing I could think of, at the time, was to have the game played iteratively. I needed to first decide how the game would force the students into a position where they would need to make and test claims based on limited evidence, so that both the structural and process components of MR would be engaged. This last point was the most difficult to deal with, but I eventually decided that a way to achieve this might be to hide the rules of the game from the students. At the time, I'll admit that this was an intuitive decision. Although I expected students to use their MR skills during the activity, I do not believe I fully realized how crucial this final decision would become until the activity was finished (I will circle back to this idea in more detail later). For clarity, I note now that I have taught this course several times and used, essentially, the same activity each time.

The Search

While creating a game seemed interesting, finding one would save me significant time. And while finding some options for games took more time than I anticipated, I eventually came across a book by Ben Orlin (2022) entitled *Mathematical games with bad drawings*, which is the following up to his bestselling book, *Math with Bad Drawings* (Orlin, 2018). I began to peruse the various games and, while it did take some time, I felt lucky to find a game known as Love and Marriage, which Orlin (2022) indicates was popularized by James Ernest, who has been working in game design since the early '90s. Satisfied that I could achieve what I wanted through this medium, I decided that I would use the shell of this game and modify it, as needed, to fit what I was trying to do. I began to shift my focus more directly towards how the students would experience both the structural and process components of MR. I needed to ensure that my version of Love and Marriage would have the students go through these two components as they played.

Details, Rules, and Structures

Students would select a laminated index card that featured a single number ranging from 1 to 75, though I will only introduce the numbers that corresponded to the number of students enrolled in the course. Once a student selected a card, they would be told that the goal was to earn the highest score they could. The procedures of the game were then detailed as follows. The students were told that they would partner up with one other student, and then both students would come to the front of the classroom to receive their score. I had created a simple Excel spreadsheet to determine the scores for each student based on the numbers on the index cards that they were holding. The scores would then be verbally given to each student as a numerical value, without any other information or context provided. Students were only told that three specific rules governed the assignment of their scores and that the numbers they were carrying played a part in their score determination. They were not told anything else about what the rules of the game were or how the score was calculated. Once the pair of students had received their scores, they were told to begin discussing with their partner what they believed the rules of the game were based on the evidence they had collected (i.e., with only their card numbers and their scores as evidence to use for generating conjectures). The student pairs were not allowed to discuss their scores with other groups. After all the students had been assigned scores, a few minutes of discussion ensued before the student pairs were broken apart and students were instructed to select a different partner in the next round. The difference between rounds was that any knowledge students had gained about the relationship between the rules and their scores from earlier rounds was preserved.

The score was determined according to some simplistic rules that turned into mathematical parameters in a formula. The rules were that: 1) the lower number of the pair was always given a higher score, 2) the pairs that approached first were given a higher score than pairs that took their time, and 3) pairs with numbers that were close together scored higher than numbers that were further apart. These rules dictated parameters that would be placed into a formula, with the first element being a *starting constant*, with higher starting constants being given to pairs of students who approached first, and then this number would be manipulated by the other rules. For example, I could have selected a starting constant of 110, then had the starting constant decrease by three for each subsequent pair of students (so Pair 1 receives 110, Pair 2 receives 107, Pair 3 receives 104, Pair 4 receives 101, and so on), eventually entering negative numbers if needed based on class size. The initial number and the decrease for each pair were arbitrary decisions, but common across iterations of the game. The *mathematical difference* between the pair of numbers on the index cards held by the students was also calculated and, to ensure it was always positive, its absolute value was taken. This value then became a divisor for the starting constant, which ensured that the mathematical difference parameter affected the scores significantly. Finally, a number called the *modifier constant* was subtracted from the score for the higher number of the pair, while the lower number had the same constant number added. This would always result in the same total difference between scores and would always assign a higher score to the lower number in the student pair. I set this modifier constant to have an arbitrary value of 10% of the starting constant so, in line with the example I gave earlier, it would have been 11 (10% of 110). The formula I used breaks down to:

$$\text{Individual Score} = \frac{\text{Starting Constant}}{|\text{Difference between Index Card Numbers}|} \pm \text{Modifier Constant}$$

For clarity, this formula could have been replaced with almost any other arrangement (instructor's choice) that served the initial rules of the game.

Iterative Reverse Engineering

Having run this game multiple times now in my first class of a semester for this course, the discussions and behavior of the students remain very similar across the rounds. In the initial round, students did not know any of the rules or how their score would be determined, and so simply matched up with their friends, someone physically sitting close to them, or whoever had the courage to ask them to join a group first. In the later rounds of the game, armed with the knowledge gained and hypotheses gleaned from discussions in previous rounds, the students began to partner up more purposefully as they attempted to tease more mathematical clues about the relationship governing the scores they were given as outputs based on the input values. Some interesting choices were being made; students began to pair up according to numbers like 37 and 73, hoping to get additional information about the rules from selected numbers that contained the same digits in different orders. Other students focused on multiples, pairing 5 and 15 or 7 and 14, while others began to close the gap between their numbers, like 40 and 41. Still other students began to widen the gap between their numbers; for example, 2 and 60, to test how intentionally widening the gap would influence their scores. Other students paired up seemingly randomly, but the number of student pairs with random pairing always seemed to decrease in later rounds. The students knew we would play more rounds, so they were not worried about potentially lowering their score to an abysmal level in one round if it helped them determine the rules of the game. Scores were given on a round-by-round basis and were not cumulative in any way. While scores were given verbally, some students chose to write them down for inspection. After about four to five rounds, many students had figured out at least one or two of the rules, and the class was instructed to work together to assure everyone the highest possible scores in the final round of the game. Students who felt confident that they could justify why they had determined certain rules began helping peers who may have had less evidence to work with.

Earlier, I mentioned that concealing the rules from the students was an intuitive decision. After reflecting on how the activity went the first time I conducted it, the process of having students reverse engineer the rules of a game using MR skills has been fascinating to observe. Friedman (2021) has indicated that reverse engineering forces us to analyze a hidden structure to reveal how an idea translates into a high-quality product or activity. When considering the elements that Jeannotte and Kieran (2017) noted as core parts of MR, it appears to be the case that pedagogical situations in which students need to reverse engineer the rules of a game are a very promising avenue worth exploring, in teaching practice as well as in research.

Future Implications

Can we definitively say that the teaching method detailed here develops MR? At this point, no. I have offered some anecdotal evidence. However, with different groups of students making similar choices over several semesters, it seems to be a plausible teaching strategy for exploring MR. Barbero et al. (2020) were able to example the reasoning path used by students as they engaged in a game (3D-Tic-Tac-Toe), by cataloguing student actions, describing a phenomenon that they referred to as "mathematical backward reasoning" (p. 1). Furthermore, recent research from Liu et al. (2023) suggests that reverse engineering pedagogy had positive effects on the development of primary school students' computational thinking skills. Clearly, then, this type of pedagogy can have a cognitive impact. However, it is not clear if MR skills will benefit in the same way as computational thinking skills. By combining these findings with the support provided by this anecdotal evidence, perhaps further resources could be devoted towards the study of MR using reverse engineering pedagogy.

I suspect similar findings would occur if activities based on the game/game structure I described in this Note were repeated across several classrooms within a more formal research structure. The strategy could then be broken down by the pre-service teacher educators to reveal a pedagogical technique that they could carry forward to use with students in the K-12 system. In any situation where students are given both a game or activity with numerical inputs and outputs along with an unclear process on how they are connected, students must attempt to reverse engineer the conditions for success by using iterations of the game, analysis of patterns, deductive/inductive reasoning, and discussion with their peers. I certainly plan to explore this teaching strategy more in different ways with my future students. Who knows? Pre-service teachers might even have fun while learning about mathematical reasoning!

NOTES

1. For clarity, my definitions of the terms (data, claim, and warrant) originate from Toulmin's model of argumentation (see Karbach, 1987), where data is the evidence, the claim is the premise, and the warrant is the justification for the data supporting the claim.

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